Tutorial - 3

Electrostatics

Chapter 4: Electrostatic Fields – Part I

Summary

1. The two fundamental laws for electrostatic fields (Coulomb's and Gauss's) are presented in this chapter. Coulomb's law of force states that

$$\mathbf{F} = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^2}\mathbf{a}_R$$

2. Based on Coulomb's law, we define the electric field intensity E as the force per unit charge; that is,

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o R^2}\mathbf{a}_R = \frac{Q\,\mathbf{R}}{4\pi\varepsilon_o R^3} \quad \text{(point charge only)}$$

3. For a continuous charge distribution, the total charge is given by

$$Q = \int_L \rho_L\, dl \qquad \text{for line charge}$$

$$Q = \int_S \rho_S\, dS \qquad \text{for surface charge}$$

$$Q = \int_v \rho_v\, dv \qquad \text{for volume charge}$$

The E field due to a continuous charge distribution is obtained from the formula for point charge by replacing Q with $dQ = \rho_L\, dl$, $dQ = \rho_S\, dS$ or $dQ = \rho_v\, dv$ and integrating over the line, surface, or volume, respectively.

4. For an infinite line charge,

$$\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_o\rho}\mathbf{a}_\rho$$

and for an infinite sheet of charge,

$$\mathbf{E} = \frac{\rho_S}{2\varepsilon_o}\mathbf{a}_n$$

5. The electric flux density D is related to the electric field intensity (in free space) as

$$\mathbf{D} = \varepsilon_o\mathbf{E}$$

The electric flux through a surface S is

$$\Psi = \int_S \mathbf{D}\cdot d\mathbf{S}$$

6. Gauss's law states that the net electric flux penetrating a closed surface is equal to the total charge enclosed, that is, $\Psi = Q_{enc}$. Hence,

$$\Psi = \oint_S \mathbf{D}\cdot d\mathbf{S} = Q_{enc} = \int_v \rho_v\, dv$$

or

$$\rho_v = \nabla\cdot\mathbf{D} \quad \text{(first Maxwell equation to be derived)}$$

When charge distribution is symmetric, so that a Gaussian surface (where $\mathbf{D} = D_n\mathbf{a}_n$ is constant) can be found, Gauss's law is useful in determining D; that is,

$$D_n \oint_S dS = Q_{enc} \quad \text{or} \quad D_n = \frac{Q_{enc}}{S}$$

Textbook Practice Exercises

PRACTICE EXERCISE 4.1

Point charges 5 nC and −2 nC are located at (2, 0, 4) and (−3, 0, 5), respectively.

(a) Determine the force on a 1 nC point charge located at (1, −3, 7).
(b) Find the electric field E at (1, −3, 7).

Answer: (a) $-1.004\mathbf{a}_x - 1.284\mathbf{a}_y + 1.4\mathbf{a}_z$ nN.
(b) $-1.004\mathbf{a}_x - 1.284\mathbf{a}_y + 1.4\mathbf{a}_z$ V/m.

P. E. 4.1

$\vec{F} = Q\vec{E}$

$$(a)\ \boldsymbol{F} = \frac{1\times10^{-9}}{4\pi\left(\frac{10^{-9}}{36\pi}\right)}\left[\frac{5\times10^{-9}[(1,-3,7)-(2,0,4)]}{[(1,-3,7)-(2,0,4)]^3} + \frac{(-2\times10^{-9})[(1,-3,7)-(-3,0,5)]}{[(1,-3,7)-(-3,0,5)]^3}\right]$$

$$= \left[\frac{45(-1,-3,3)}{19^{3/2}} - \frac{18(4,-3,2)}{29^{3/2}}\right] \text{ nN}$$

$$= \underline{\underline{-1.004\boldsymbol{a}_x - 1.284\boldsymbol{a}_y + 1.4\boldsymbol{a}_z \text{ nN}}}$$

$$(b)\quad \boldsymbol{E} = \frac{\boldsymbol{F}}{Q} = \underline{\underline{-1.004\boldsymbol{a}_x - 1.284\boldsymbol{a}_y + 1.4\boldsymbol{a}_z \text{ V/m}}}$$

PRACTICE EXERCISE 4.2

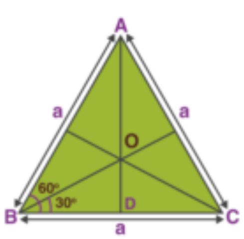
Three identical small spheres of mass m are suspended from a common point by threads of negligible masses and equal length ℓ. A charge Q is divided equally among the spheres, and they come to equilibrium at the corners of a horizontal equilateral triangle whose sides are d. Show that

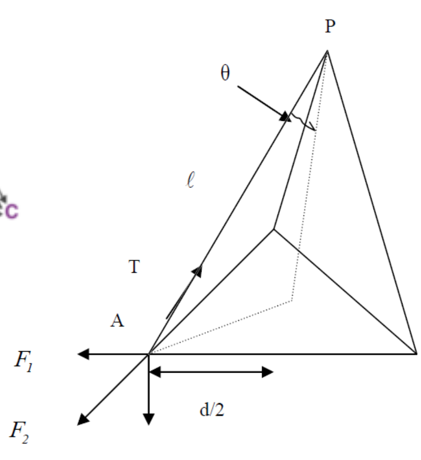
$$Q^2 = 12\pi\varepsilon_o mgd^3\left[\ell^2 - \frac{d^2}{3}\right]^{-1/2}$$

where g = acceleration due to gravity.

Answer: Proof.

P. E. 4.2
Let q be the charge on each sphere, i.e. q=Q/3. The free body diagram below helps us to establish the relationship between various forces.





At point A,

$$T\sin\theta\cos 30^\circ = F_1 + F_2\cos 60^\circ$$

$$= \frac{q^2}{4\pi\varepsilon_0 d^2} + \frac{q^2}{4\pi\varepsilon_0 d^2}\left(\frac{1}{2}\right)$$

$$= \frac{3q^2}{8\pi\varepsilon_0 d^2}$$

$$T\cos\theta = mg$$

Hence, $$\tan\theta\cos 30^\circ = \frac{3q^2}{8\pi\varepsilon_0 d^2 mg}$$

But $$\sin\theta = \frac{h}{l} = \frac{d}{\sqrt{3}\,l}\qquad \tan\theta = \frac{\frac{d}{\sqrt{3}}}{\sqrt{l^2 - \frac{d^2}{3}}}$$

Thus, $$\frac{\frac{d}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{l^2 - \frac{d^2}{3}}} = \frac{3q^2}{8\pi\varepsilon_0 d^2 mg}$$

or $$q^2 = \frac{4\pi\varepsilon_0 d^3 mg}{3\sqrt{l^2 - \frac{d^2}{3}}}$$

but $q = \dfrac{Q}{3} \longrightarrow q^2 = \dfrac{Q}{9}$. Hence,

$$Q^2 = \underline{\underline{\frac{12\pi\varepsilon_0 d^3 mg}{\sqrt{l^2 - \frac{d^2}{3}}}}}$$

PRACTICE EXERCISE 4.3

An ion rocket emits positive cesium ions from a wedge-shaped electrode into the region described by $x > |y|$. The electric field is $\mathbf{E} = -400\mathbf{a}_x + 200\mathbf{a}_y$ kV/m. The ions have single electronic charges $e = -1.6019 \times 10^{-19}$ C and mass $m = 2.22 \times 10^{-25}$ kg, and they travel in a vacuum with zero initial velocity. If the emission is confined to -40 cm $< y < 40$ cm, find the largest value of x that can be reached.

d

Answer: 0.8 m.

$\vec{E} = -400\,\hat{a}_x + 200\,\hat{a}_y$ kV/m

$\vec{F} = Q\vec{E} = mg = ma$

$Q\vec{E} = ma$

$\bar{e}E = ma$

$a = \dfrac{\bar{e}\vec{E}}{m}$

P.E. 4.3

$$e\bar{E} = m\frac{d^2\bar{l}}{dt^2}$$

$$eE_0(-2\bar{a}_x + \bar{a}_y) = m\left(\frac{d^2x}{dt^2}\bar{a}_x + \frac{d^2y}{dt^2}\bar{a}_y + \frac{d^2z}{dt^2}\bar{a}_z\right)$$

where $E_0 = 200$ kV / m

$$\frac{d^2z}{dt^2} = 0 \longrightarrow z = ct + c_2$$

$$m\frac{d^2x}{dt^2} = -2eE_0 \longrightarrow x = \frac{-2eE_0t^2}{2m} + c_3t + c_4$$

$$m\frac{d^2y}{dt^2} = eE_0 \longrightarrow y = \frac{eE_0t^2}{2m} + c_5t + c_6$$

At $t = 0$, $(x, y, z) = (0, 0, 0)$ $\quad c_1 = 0 = c_4 = c_6$

Also, $\left(\dfrac{dx}{dt}, \dfrac{dy}{dt}, \dfrac{dz}{dt}\right) = (0, 0, 0)$

At $t = 0 \longrightarrow c_1 = 0 = c_3 = c_5$

Hence, $\quad (x, y) = \dfrac{eE_0t^2}{2m}(-2, 1)$

i.e. $2|y| = |x|$

Thus the largest value of is

80 cm = <u>0.8 m</u>

$a = \dfrac{e}{m} < \boxed{-400\times10^3}\,\hat{a}_x + 200\times10^3\,\hat{a}_y$

$a_x = -2{,}887\times10^{11}$ m/s²

$a_y = 1{,}4444\times10^{11}$ m/s²

$d = \cancel{v_i t} + \frac{1}{2}at^2$

-40 or 40

$\boxed{y} = \frac{1}{2}a_y t^2$

$t = 2{,}35\times10^{-6}$

$\boxed{x} = \frac{1}{2}a_x \boxed{t^2}$

$x = 0{,}8$ m

PRACTICE EXERCISE 4.4

A circular disk of radius a is uniformly charged with ρ_S C/m². The disk lies on the $z = 0$ plane with its axis along the z-axis.

(a) Show that at point $(0, 0, h)$

$$\mathbf{E} = \frac{\rho_S}{2\varepsilon_o}\left\{1 - \frac{h}{[h^2 + a^2]^{1/2}}\right\}\mathbf{a}_z$$

(b) From this, derive the E field due to an infinite sheet of charge on the $z = 0$ plane.

(c) If $a \ll h$, show that E is similar to the field due to a point charge.

Answer: (a) Proof, (b) $\dfrac{\rho_S}{2\varepsilon_0}\mathbf{a}_z$, (c) Proof.

P.E. 4.4

(a)

Consider an element of area dS of the disk.

The contribution due to $dS = \rho d\phi d\rho$ is

$$dE = \frac{\rho_s dS}{4\pi\varepsilon_0 r^2} = \frac{\rho_s dS}{4\pi\varepsilon_0(\rho^2 + h^2)}$$

The sum of the contribution along ρ gives zero.

$$E_z = \frac{\rho_s}{4\pi\,\varepsilon_0}\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\frac{h\rho\,d\rho\,d\phi}{(\rho^2+h^2)^{3/2}} = \frac{h\,\rho_s}{2\,\varepsilon_0}\int_{\rho=0}^{a}\frac{\rho\,d\rho}{(\rho^2+h^2)^{3/2}}$$

$$= \frac{h\,\rho_s}{4\,\varepsilon_0}\int_0^a(\rho^2+h^2)^{-3/2}d(\rho^2) = \frac{h\rho_s}{4\varepsilon_0}(-2(\rho^2+h^2)^{-1/2}\Big|_0^a$$

$$= \frac{\rho_s}{2\varepsilon_0}[1 - \frac{h}{(h^2+a^2)^{1/2}}]$$

(b)

As $a \longrightarrow \infty$,

$$\boldsymbol{E} = \frac{\rho_s}{2\varepsilon_0}\boldsymbol{a}_z$$

(c) Let us recall that if $a/h << 1$ then $(1+a/h)^n$ can be approximated by $(1+na/h)$. Thus the expression for E_z from (a) can be modified for $a<<h$ as follows.

$$E_z = \frac{\rho_s}{2\varepsilon_o}\left[1 - \frac{1}{\sqrt{1+\frac{a^2}{h^2}}}\right] = \frac{\rho_s}{2\varepsilon_o}\left[1 - \left(1+\frac{a^2}{h^2}\right)^{-\frac{1}{2}}\right] \xrightarrow{a\to 0,\ but\ \rho_s\pi a^2 = Q} \frac{\rho_s}{2\varepsilon_o}\left[\frac{a^2}{2h^2}\right]$$

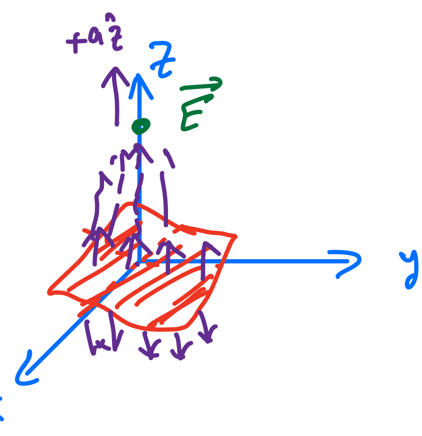
$$= \frac{\rho_s}{2\varepsilon_o}\left[\frac{\pi a^2}{2\pi h^2}\right] = \frac{Q}{4\pi\varepsilon_o h^2}$$

This is in keeping with original Coulomb's law.

PRACTICE EXERCISE 4.5

A square plate described by $-2 \le x \le 2$, $-2 \le y \le 2$, $z = 0$ carries a charge $12\,|y|$ mC/m². Find the total charge on the plate and the electric field intensity at $(0, 0, 10)$.

Answer: 192 mC, 16.6 $\mathbf{a}_z$ MV/m.



P. E. 4.5

$$Q_S = \int \rho_S dS = \int_{-2}^{2}\int_{-2}^{2} 12|y|\,dx\,dy$$

$$= 12(4)\int_0^2 2\,y\,dy = \underline{192}\ \text{mC}$$

$$\boldsymbol{E} = \int\frac{\rho_s dS}{4\pi\varepsilon r^2}\boldsymbol{a}_r = \boxed{\int\frac{\rho_s dS\,|\boldsymbol{r}-\boldsymbol{r}'|}{4\pi\varepsilon_o\,|\boldsymbol{r}-\boldsymbol{r}'|^3}}$$

where $\boldsymbol{r}-\boldsymbol{r}' = (0,0,10)-(x,y,z) = (-x,-y,10)$.

$$Q = \iint \rho_s\, ds$$

$$= \int_{-2}^{2}\int_{-2}^{2} \boxed{12\times10^{-3}}\,|y|\,dx\,dy$$

$$= 12\times10^{-3}\,(4)\int_{-2}^{2}\boxed{|y|}\,dy$$

$$= 192\ \text{mC}$$

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$$\vec{E} = \int \frac{\rho_s\, ds}{4\pi\varepsilon r^2}\, \boxed{\hat{a}_r}$$

desk → infinite sheet

$$\frac{\langle r-r' \rangle}{|r-r'|}$$

$$E = \int_{x=-2}^{2}\int_{y=-2}^{2} \frac{12\,|y|10^{-3}(-x,-y,10)}{4\pi(\frac{10^{-9}}{36\pi})(x^2+y^2+100)^{3/2}}$$

$$= 108(10^6)\Big[\int_{-2}^{2}|y|\int_{-2}^{2}\frac{-x\,dx\,dy\;\boldsymbol{a}_x}{(x^2+y^2+100)^{3/2}} + \int_{-2}^{2}\int_{-2}^{2}\frac{-y|y|\;dy\,dx\,\boldsymbol{a}_y}{(x^2+y^2+100)^{3/2}}$$

$$+10\,\boldsymbol{a}_z\int_{-2}^{2}\int_{-2}^{2}\frac{|y|\,dx\,dy}{(x^2+y^2+100)^{3/2}}\Big]$$

$$\vec{E} = \iint \frac{12|y|}{4\pi\varepsilon_0}\,\frac{\langle r-r' \rangle}{|r-r'|}\,dx\,dy$$

$$E = 108(10^7)\,\boldsymbol{a}_z\int_{-2}^{2}\Big[2\int_{0}^{2}\frac{\frac{1}{2}d(y^2)}{(x^2+y^2+100)^{3/2}}\Big]dx$$

$$= -216(10^7)\,\boldsymbol{a}_z\int_{-2}^{2}\Big[\frac{1}{(x^2+104)^{1/2}} - \frac{1}{(x^2+100)^{1/2}}\Big]dx$$

$$|r-r'| = (\boxed{0}, \boxed{0}, 10) - (\boxed{?}, ?, 0)$$

مكان الراصد ← r

المصدر ← r'

أعوض اللي يتغير

-5, 3, ρ, -5, 3

$$= -216(10^7)\,\boldsymbol{a}_z \ln\left|\frac{x+\sqrt{x^2+104}}{x+\sqrt{x^2+100}}\right|_{-2}^{2}$$

$$= -216(10^7)\,\boldsymbol{a}_z\left(\ln\left(\frac{2+\sqrt{108}}{2+\sqrt{104}}\right) - \ln\left(\frac{-2+\sqrt{108}}{-2+\sqrt{104}}\right)\right)$$

$$= -216(10^7)\,\boldsymbol{a}_z(-7.6202\,(10^{-3}))$$

$$E = \underline{16.46\;\boldsymbol{a}_z\;\text{MV/m}}$$

$$r - r' = (0, 0, 10\hat{a}_z)$$

$$|r-r'| = 10$$

PRACTICE EXERCISE 4.6

In Example 4.6 if the line $x = 0, z = 2$ is rotated through 90° about the point $(0, 2, 2)$ so that it becomes $x = 0, y = 2$, find E at $(1, 1, -1)$.

Answer: $-282.7\mathbf{a}_x + 565.5\mathbf{a}_y$ V/m.

$$E = \int_{-2}^{2}\int_{-2}^{2} \frac{12\times10^{-3}\,|y|}{4\pi\varepsilon_0\,(10)^3}\,10\,\hat{a}_z\;dx\,dy$$

$$\int_{-2}^{2} dx = 4$$

$\boldsymbol{E}_1$ and $\mathbf{E}_2$ remain the same as in Example 4.6.

$$\boldsymbol{E}_3 = \frac{\rho_L}{2\pi\varepsilon_0\rho}\boldsymbol{a}_\rho$$

This expression, which represents the field due to a line charge, is modified as follows. To get $\boldsymbol{a}_\rho$, consider the $z=-1$ plane. $\rho = \sqrt{2}$

$$\boldsymbol{a}_\rho = \boldsymbol{a}_x\cos 45° - \boldsymbol{a}_y\sin 45°$$

$$= \frac{1}{\sqrt{2}}(\boldsymbol{a}_x - \boldsymbol{a}_y)$$

$$\boldsymbol{E}_3 = \frac{10(10^{-9})}{2\pi(\frac{10^{-9}}{36\pi})}\frac{1}{2}(\boldsymbol{a}_x - \boldsymbol{a}_y)$$

$$= 90\,\pi\,(\boldsymbol{a}_x - \boldsymbol{a}_y). \quad \text{Hence,}$$

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 + \boldsymbol{E}_3$$

$$= -180\,\pi\boldsymbol{a}_x + 270\,\pi\boldsymbol{a}_y + 90\,\pi\boldsymbol{a}_x - 90\,\pi\boldsymbol{a}_y$$

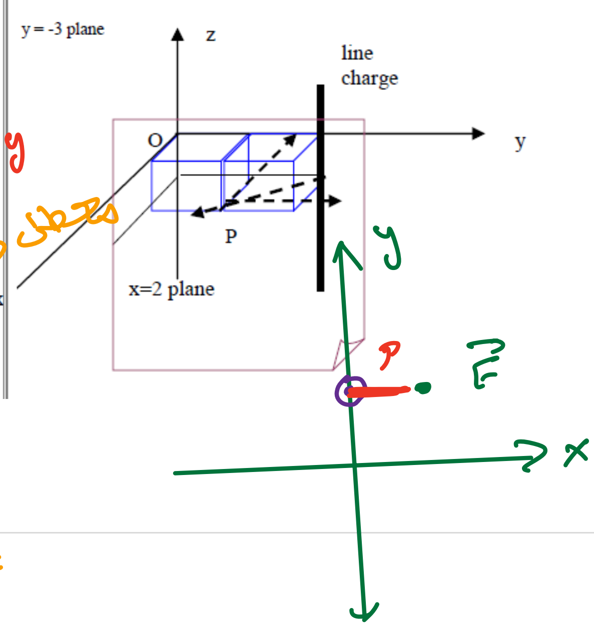
$$= \underline{\underline{-282.7\,\boldsymbol{a}_x + 565.5\boldsymbol{a}_y \quad \text{V/m}}}$$

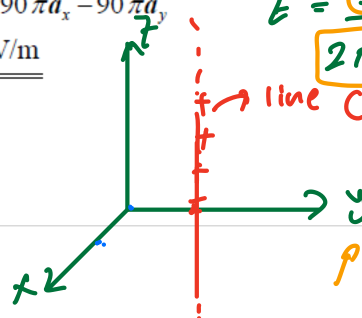
$\vec{E}$: point, line, sheet

line: finite, infinite, ring

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\,\hat{a}_\rho$$

P.E. 4.6





PRACTICE EXERCISE 4.7

A point charge of 30 nC is located at the origin, while plane $y = 3$ carries charge 10 nC/m². Find **D** at $(0, 4, 3)$.

Answer: $5.076\mathbf{a}_y + 0.0573\mathbf{a}_z$ nC/m².

$P = (1, 1, \boxed{-1}) = (0, 2, \boxed{?})$

$\vec{P} = (1, -1) = \langle \hat{a}_x - \hat{a}_y \rangle$

$|\vec{P}| = \sqrt{2}$

$\hat{a}_P = \boxed{\dfrac{\langle \hat{a}_x - \hat{a}_y \rangle}{\sqrt{2}}}$

$\vec{E} = \dfrac{\rho_L}{2\pi\varepsilon_0 \rho}\hat{a}_\rho = \dfrac{10\times10^{-9}}{2\pi\varepsilon_0\sqrt{2}}$

P.E. 4.7

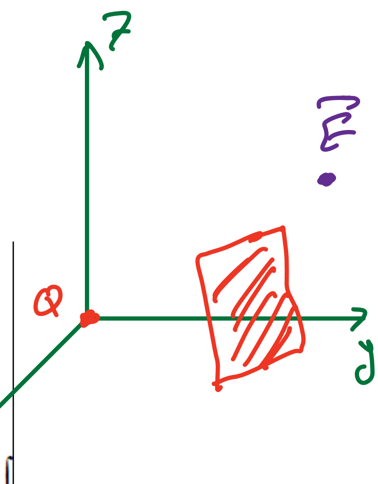
$$\boldsymbol{D} = \boldsymbol{D}_Q + \boldsymbol{D}_\rho = \frac{Q}{4\pi r^2}\boldsymbol{a}_r + \frac{\rho_s}{2}\boldsymbol{a}_n$$

$$= \frac{30\times10^{-9}}{4\pi(5)^2}\,\frac{[(0,4,3)-(0,0,0)]}{5} + \frac{10\times10^{-9}}{2}\boldsymbol{a}_y$$

$$= \frac{30}{500\pi}(0,4,3) + 5\boldsymbol{a}_y \ \text{nC/m}^2$$

$$= \underline{\underline{5.076\boldsymbol{a}_y + 0.0573\boldsymbol{a}_z \ \text{nC/m}^2}}$$

4.7



PRACTICE EXERCISE 4.8

If $\mathbf{D} = (2y^2 + z)\mathbf{a}_x + 4xy\mathbf{a}_y + x\mathbf{a}_z$ C/m², find

(a) The volume charge density at $(-1, 0, 3)$
(b) The flux through the cube defined by $0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1$
(c) The total charge enclosed by the cube

Answer: (a) -4 C/m³, (b) 2 C, (c) 2 C.

P.E. 4.8

(a) $\rho_v = \nabla \bullet \boldsymbol{D} = 4x$

$\rho_v(-1,0,3) = \underline{\underline{-4 \ \text{C/m}^3}}$

(b) $\Psi = \oint \boldsymbol{D}\bullet d\boldsymbol{S} = \iint \boldsymbol{D}\bullet d\boldsymbol{S}\Big|_{x=0} + \iint \boldsymbol{D}\bullet d\boldsymbol{S}\Big|_{x=1} + \iint \boldsymbol{D}\bullet d\boldsymbol{S}\Big|_{y=0} + \iint \boldsymbol{D}\bullet d\boldsymbol{S}\Big|_{y=1} + \iint \boldsymbol{D}\bullet d\boldsymbol{S}\Big|_{z=0} + \int$

$$= -\int_0^1\!\!\int_0^1 (2y^2+z)\,dydz + \int_0^1\!\!\int_0^1 (2y^2+z)\,dydz + \int_0^1\!\!\int_0^1 4\,\mathrm{x}(1)\,dxdz - \int_0^1\!\!\int_0^1 (\mathrm{x})\,dxdz + \int_0^1\!\!\int_0^1 (\mathrm{x})\,dxdz$$

$$= 4\int_0^1\!\!\int_0^1 \mathrm{x}\,dxdz = 4(1/2)(1) = 2C$$

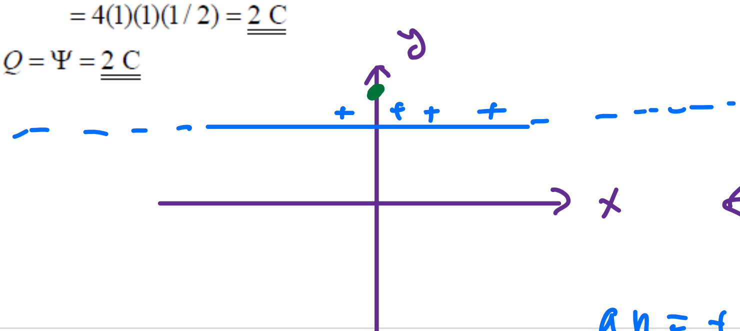
(c) $\Psi = Q = \int \rho_v dv = \int_0^1\!\!\int_0^1\!\!\int_0^1 4x\,dxdydz$

$= 4(1)(1)(1/2) = \underline{\underline{2\ \text{C}}}$

$Q = \Psi = \underline{\underline{2\ \text{C}}}$

$E_Q = \dfrac{Q}{4\pi\varepsilon_0 |r - r'|} \cdot \langle r - r' \rangle$

$E_Q = $



$E_{sheet} = \dfrac{\boxed{\rho_s}}{\boxed{2\varepsilon_0}}\boxed{\hat{a}_n}$

$a_n = \pm$

PRACTICE EXERCISE 4.9

A charge distribution in free space has $\rho_v = 2r\ \text{nC/m}^3$ for $0 \leq r \leq 10$ m and zero otherwise. Determine E at $r = 2$ m and $r = 12$ m.

Answer: $226\mathbf{a}_r$ V/m, $3.927\mathbf{a}_r$ kV/m.

P.E. 4.9

$$Q = \int \rho v dv = \psi = \oint \boldsymbol{D} \bullet d\boldsymbol{S}$$

For $0 \leq r \leq 10$,

$$D_r(4\pi r^2) = \iiint 2r\ (r^2)\sin\theta\, d\theta\, dr\, d\phi$$

$$D_r(4\pi r^2) = 4\pi\left(\frac{2r^4}{4}\Big|_0^r\right) = 2\pi r^4$$

$$D_r = \frac{r^2}{2} \qquad \boldsymbol{E} = \frac{r^2}{2\varepsilon_0}\boldsymbol{a}_r \ \text{nV/m}$$

$$\boldsymbol{E}(r=2) = \frac{4(10^{-9})}{2(\frac{10^{-9}}{36\pi})}\boldsymbol{a}_r = 72\pi\boldsymbol{a}_r = \underline{\underline{226\ \boldsymbol{a}_r\ \text{V/m}}}$$

For $r \geq 10$,

$$D_r(4\pi r^2) = 2\pi r_0^{\,4}, \qquad r_0 = 10\text{m}$$

$$D_r = \frac{r_0^{\,4}}{2r^2} \longrightarrow \boldsymbol{E} = \frac{r_0^{\,4}}{2\varepsilon_0 r^2}\boldsymbol{a}_r \ \text{nV/m}$$

$$\boldsymbol{E}(r=12) = \frac{10^4(10^{-9})}{2(\frac{10^{-9}}{36\pi})(144)}\boldsymbol{a}_r = 1250\pi\boldsymbol{a}_r$$

$$= \underline{\underline{3.927\boldsymbol{a}_r\ \text{kV/m}}}$$

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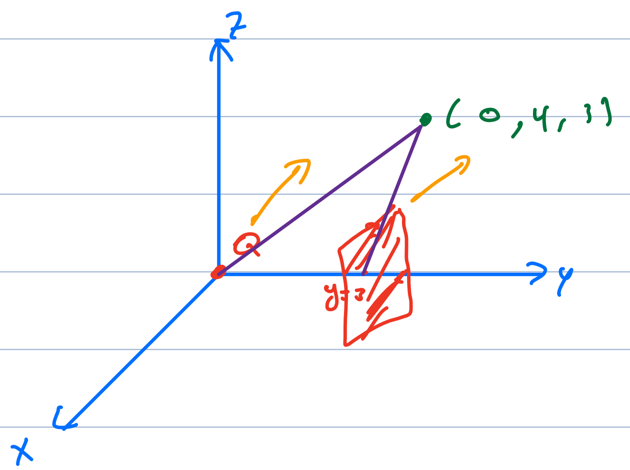
PRACTICE EXERCISE 4.7

A point charge of 30 nC is located at the origin, while plane $y = 3$ carries charge $10\ \text{nC/m}^2$. Find **D** at $(0, 4, 3)$.

Answer: $5.076\mathbf{a}_y + 0.0573\mathbf{a}_z\ \text{nC/m}^2$.

$$\vec{E}_{point} = \frac{Q}{4\pi\varepsilon_0 r^2} \qquad \vec{D} = \varepsilon_0 E \qquad \vec{D}_{point} = \frac{Q}{4\pi r^2}\hat{a_r}$$

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0}\boxed{\hat{a_n}} \qquad \vec{D} = \frac{\rho_s}{2}\hat{a_n}$$



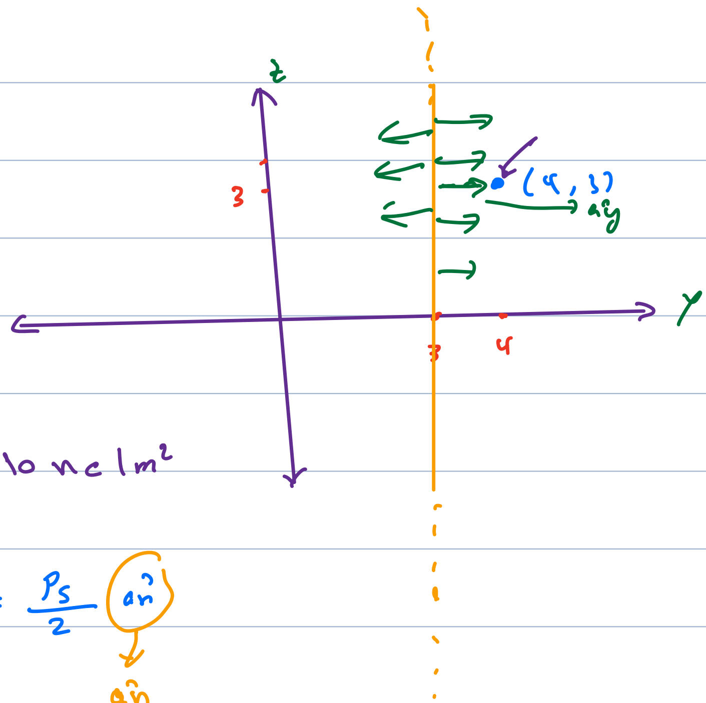
$$\vec{D} = \frac{Q}{4\pi r^2}\hat{a_r}$$

$$r = 5 \qquad \hat{a_r} = \frac{<0,4,3>}{5}$$

$$r - r' = (0,4,3) - (0,0,0) = (0,4,3)$$

$$|r - r'| = \sqrt{4^2 + 3^2} = 5$$

$$\vec{D}_a = \frac{30\times10^{-9}}{4\pi(5)^2} \cdot \frac{<4\hat{a_y} + 3\hat{a_z}>}{5}$$



$\rho_s = 10\ nc/m^2$

$$\vec{D} = \frac{\rho_s}{2}\, \hat{a}_n$$

$\hat{a}_y$

$$\vec{D}_{\rho_s} = \frac{10\times10^{-9}}{2}\, \hat{a}_y$$

$$\vec{D}_T = \vec{D}_Q + \vec{D}_{\rho_s} = \frac{30\times10^{-9}}{4\pi\,(5)^2} \cdot \frac{\langle 4\hat{a}_y + 3\hat{a}_z \rangle}{5} + \frac{10\times10^{-9}}{2}\, \hat{a}_y$$

PRACTICE EXERCISE 4.8

If $\mathbf{D} = \boxed{(2y^2 + z)}\mathbf{a}_x + \boxed{4xy\mathbf{a}_y} + \boxed{x\mathbf{a}_z}$ C/m², find

$\nabla \cdot \mathbf{D}$

(a) The volume charge density at $(-1, 0, 3)$
(b) The flux through the cube defined by $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$
(c) The total charge enclosed by the cube

Answer: (a) $\boxed{-4 \text{ C/m}^3}$, (b) 2 C, (c) 2 C.

a) ρ_v

$$\rho_v = \nabla \cdot \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\rho_v = 0 + 4x + 0 = 4x = -4 \text{ C/m}^3$$

b) $$\boxed{\oint \vec{D} \cdot d\vec{s}} = \Psi_{net} = Q_{net}$$

$$\Psi = \iiint \rho_v \, dv \qquad dv = dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 4x \, dx\,dy\,dz$$

$$\Psi = 2 \text{ C}$$

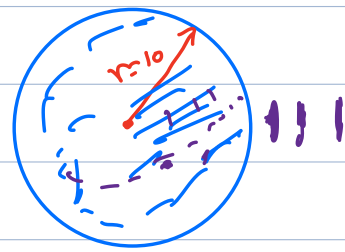
c) $$Q_{net} = \iiint \rho_v \, dv = 2 \text{ C}$$

PRACTICE EXERCISE 4.9

A charge distribution in free space has $\rho_v = 2r\ \text{nC/m}^3$ for $0 \le r \le 10$ m and zero otherwise. Determine **E** at $r = 2$ m and $r = 12$ m.

Answer: $226\mathbf{a}_r$ V/m, $3.927\mathbf{a}_r$ kV/m.

$$\oint \vec{D} \cdot d\vec{s} = Q_{net} = \Psi_{net}$$



$$\oint \vec{D} \cdot d\vec{s} = Q_{net}$$

$r < 10$

$$\int_0^{2\pi}\int_0^{\pi} \vec{D}\, r^2 \sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi}\int_0^{r} (2)r\; r^2 \underbrace{\sin\theta\, d\theta\, d\phi}\, dr$$

$$\vec{D}\, 4\pi r^2 = 4\pi(2)\int_0^r r^3\, dr$$

$$\vec{D}\, 4\pi r^2 = 8\pi \left[\frac{r^4}{4}\right]_0^r$$

$$\vec{D}\, \cancel{4\pi}\, \cancel{r^2} = \cancel{2\pi}\, r^{\cancel{4}}$$

$$\vec{D} = \frac{r^2}{2}\hat{a}_r$$

$$\boxed{\vec{E} = \frac{r^2}{2\varepsilon_0}\hat{a}_r} \qquad \vec{E}_{r=2} = \frac{2^2}{2\varepsilon_0}\hat{a}_r$$

$$\int_0^{2\pi}\int_0^{\pi} \sin\theta\, d\theta\, d\phi = (2\pi)\int_0^{\pi}\sin\theta\, d\theta = 4\pi$$

$\boxed{r > 10}$

$$\oint \vec{D} \cdot ds = Q_{net}$$

$$\vec{D}\, 4\pi r^2 = \boxed{\int_0^{2\pi}\int_0^{\pi}\int_0^{10} 2r \quad r^2 \sin\theta \, dr \, d\theta \, d\phi}$$

$$\vec{D}\, 4\pi \, r^2 = 2\pi (10)^4$$

$$\vec{D} = \frac{10^4}{2 r^2} \, a_r$$

$$\vec{E} = \frac{10^4}{2 r^2 \varepsilon_0} \, a_r$$

$$\vec{E}_{r=12} = \frac{10^4}{2 (12)^2 \varepsilon_0} \, a_r$$